CCFU Proof 8

Fractal Similarity Dimension $D = 1$ for φ-Branching Tree

Given. Let $\varphi = (1+\sqrt{5})/2$, so $\varphi^2 = \varphi + 1$. A Pythagoras tree with branch scales $1/\varphi$ and $1/\varphi^2$. Assume the IFS satisfies the open set condition (no geometric overlaps between branches).

IFS equation for similarity dimension D:

$$(1/\varphi)^D + (1/\varphi^2)^D = 1.$$

Claim. $D = 1$ is the unique solution.

Proof.

$$(1/\varphi)^1 + (1/\varphi^2)^1 = \frac{1}{\varphi} + \frac{1}{\varphi^2} = \frac{\varphi+1}{\varphi^2} = \frac{\varphi^2}{\varphi^2} = 1, \qquad \blacksquare$$

where we used $\varphi + 1 = \varphi^2$.

Uniqueness. Let $g(D) = \varphi^{-D} + \varphi^{-2D}$. Then g is strictly decreasing in D (sum of decreasing exponentials). Since $g(0) = 2 > 1$ and $\lim_{D\to\infty} g(D) = 0 < 1$, the equation $g(D) = 1$ has exactly one solution. Since $g(1) = 1$, $D = 1$ is that unique solution. $\blacksquare$

Note. This proves the similarity dimension. Under the open set condition, the Hausdorff dimension equals this value. If geometric overlaps occur in a specific realization, the Hausdorff dimension may be lower.